Explicit fundamental solutions of some second order differential operators on Heisenberg groups

Isolda Cardoso
ECEN-FCEIA, Universidad Nacional de Rosario
Pellegrini 250, 2000 Rosario, Argentina
E-mail: isolda@fceia.unr.edu.ar

Linda Saal
FAMAF, Universidad Nacional de Córdoba
Ciudad Universitaria, 5000 Córdoba, Argentina
E-mail: saal@mate.uncor.edu

Abstract

Let p, q, n be natural numbers such that $p + q = n$. Let $\mathbb{F}$ be either $\mathbb{C}$, the complex numbers field, or $\mathbb{H}$, the quaternionic division algebra. We consider the Heisenberg group $N(p, q, \mathbb{F})$ defined as $N(p, q, \mathbb{F}) = \mathbb{F}^n \times \mathfrak{Im}\mathbb{F}$, with group law given by

$$(v, \zeta)(v', \zeta') = \left(v + v', \zeta + \zeta' - \frac{1}{2}\mathfrak{Im}B(v, v')\right),$$

where $B(v, w) = \sum_{j=1}^{p} v_j\overline{w_j} - \sum_{j=p+1}^{n} v_j\overline{w_j}$. Let $U(p, q, \mathbb{F})$ be the group of $n \times n$ matrices with coefficients in $\mathbb{F}$ that leave invariant the form B. In this work we compute explicit fundamental solutions of some second order differential operators on $N(p, q, \mathbb{F})$ which are canonically associated to the action of $U(p, q, \mathbb{F})$.

1 Introduction and Preliminaries

Let p, q, n be natural numbers such that $p + q = n$. Let $\mathbb{F}$ be either $\mathbb{C}$, the complex numbers field, or $\mathbb{H}$, the quaternionic division algebra. We consider the Heisenberg group $N(p, q, \mathbb{F})$ defined as $N(p, q, \mathbb{F}) = \mathbb{F}^n \times \mathfrak{Im}\mathbb{F}$, with group law given by

$$(v, \zeta)(v', \zeta') = \left(v + v', \zeta + \zeta' - \frac{1}{2}\mathfrak{Im}B(v, v')\right),$$

where $B(v, w) = \sum_{j=1}^{p} v_j\overline{w_j} - \sum_{j=p+1}^{n} v_j\overline{w_j}$. The associated Lie algebra is $\eta(p, q, \mathbb{F}) = \mathbb{F}^n \oplus \mathfrak{Im}(\mathbb{F})$, with Lie bracket given by

$$[(v, \zeta), (v', \zeta')] = (0, -\mathfrak{Im}B(v, v')).$$

2000 *Mathematics Subject Classification*: Primary 43A80; Secondary 35A08.

Key words and phrases: Heisenberg group, spherical distributions, fundamental solution.

This research was partly supported by research grants form SCyT-UNR and Secyt-UNC, and a fellowship from CONICET.

Let $\mathcal{U}(\eta(p,q,\mathbb{F}))$ be the universal envelopping algebra of $\eta(p,q,\mathbb{F})$ which we identify with the algebra of left invariant differential operators. Let $U(p,q,\mathbb{H})$ be the group of $n\times n$ matrices with coefficients in $\mathbb{F}$ that leave invariant the form B. Then $U(p,q,\mathbb{F})$ acts by automorphism on $\eta(p,q,\mathbb{F})$ by

$$g\cdot(v,\zeta)=(gv,\zeta).$$

We denote by $\mathcal{U}(\eta(p,q,\mathbb{F}))^{U(p,q,\mathbb{F})}$ the subalgebra of $\mathcal{U}(\eta(p,q,\mathbb{F}))$ of the left invariant differential operators which commute with this action. It is known that this subalgebra is generated by two operators: L and U, and a family of tempered joint eigendistributions is computed explicitly (see for example [D-M], [G-S(1)], [V]).

More precisely, if $\mathbb{F}=\mathbb{C}$ and $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,U\}$ denotes the standard basis of the Heisenberg Lie algebra with $[X_i,Y_j]=\delta_{ij}U$ and all the other brackets are zero, then

$$L=\sum_{j=1}^{p}X_j^2+Y_j^2-\sum_{j=p+1}^{n}X_j^2+Y_j^2.$$

For $\lambda\in\mathbb{R}$, $\lambda\neq 0$ and $k\in\mathbb{Z}$, $S_{\lambda,k}$ is a $U(p,q)$-invariant tempered distribution satisfying

$$\begin{array}{rcl} LS_{\lambda,k} & = & -|\lambda|(2k+p-q)S_{\lambda,k}, \\ iUS_{\lambda,k} & = & \lambda S_{\lambda,k}. \end{array}$$

This family provides us an *inversion formula*: for all f in the Schwartz space on the Heisenberg group, we have that

$$f(z,t)=\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}f*S_{\lambda,k}|\lambda|^n d\lambda,\qquad (z,t)\in N(p,q,\mathbb{C}). \tag{1.1}$$

If $\mathbb{F}=\mathbb{H}$ we take $\{X_1^0,X_1^1,X_1^2,X_1^3,\ldots,X_n^0,X_n^1,X_n^2,X_n^3,Z_1,Z_2,Z_3\}$ the canonical basis for the Lie algebra, where Z_1,Z_2,Z_3 generate the center of $\eta(p,q,\mathbb{H})$. Here,

$$\begin{aligned} L&=\sum_{r=1}^{p}\sum_{l=0}^{3}(X_r^l)^2-\sum_{r=p+1}^{n}\sum_{l=0}^{3}(X_r^l)^2, \qquad \text{and} \\ U&=\sum_{l=1}^{3}Z_l^2. \end{aligned}$$

There also exists a family of $U(p,q,\mathbb{H})-$invariant tempered distributions $\varphi_{w,k}$, $w\in\mathbb{R}^3$ y $k\in\mathbb{Z}$, such that each one of them is a joint eigendistribution of L and U:

$$\begin{aligned} L\varphi_{w,k}&=-|w|(2k+2(p-q))\varphi_{w,k}, \\ U\varphi_{w,k}&=-\lambda^2\varphi_{w,k}; \end{aligned}$$

in this case this family also provides an inversion formula: for all $f\in\mathcal{S}(N(p,q,\mathbb{H}))$ we have that

$$f(\alpha,z)=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^3}(f*\varphi_{w,k})(\alpha,z)|w|^{2n}dw,\qquad (\alpha,z)\in N(p,q,\mathbb{H}). \tag{1.2}$$

The aim of this work is to explicitly compute a fundamental solution in the classical case for the operator $\mathcal{L}_\alpha=L+i\alpha U$, where α is a complex number; and in the quaternionic case for the operator L. Recall that a fundamental solution for the differential operator $\mathcal{L}$ is a tempered distribution Φ such

that for all f in the Schwartz class, we have that $\mathcal{L}(f*\Phi)=(\mathcal{L}f)*\Phi=f*\mathcal{L}(\Phi)=f$. So if we define the operator K as $Kf=f*\Phi$, then $K\circ\mathcal{L}f=\mathcal{L}\circ Kf=f$.

From the inversion formula (1.1) it is natural to propose as a fundamental solution of $\mathcal{L}_\alpha$

$$< \Phi_\alpha, f >=\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{1}{-|\lambda|(2k+p-q-\alpha\ \mathrm{sgn}\ \lambda)} < S_{\lambda,k}, f > |\lambda|^n d\lambda, \tag{1.3}$$

for $f\in\mathcal{S}(N(p,q,\mathbb{C}))$; and from (1.2) we propose as a fundamental solution of L

$$< \Phi, f >=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^3}\frac{1}{-|w|(2k+2(p-q))} < \varphi_{w,k}, f > |w|^{2n} dw, \tag{1.4}$$

for $f\in\mathcal{S}(N(p,q,\mathbb{H}))$.

We remark that for $q=0$, $\mathbb{F}=\mathbb{C}$ we recover the fundamental solution for the operator $\mathcal{L}_\alpha$ given in [F-S], and for $q=0$, $\mathbb{F}=\mathbb{H}$ we recover Kaplan's fundamental solution for the operator L given in [K]. The case $q\neq 0$, $\alpha=0$ was obtained in [G-S(2)].

The expression of Φ_α is obtained in theorem 2.9, and for the computation we follow the method used in [G-S(2)]. In the quaternionic case, Φ is given in theorem 3.1, and for its computation we use the Radon transform in order to reduce this case to the classical one.

To describe both families of eigendistributions $\{S_{\lambda,k}\}$ and $\{\varphi_{w,k}\}$ we need to adapt a result by Tengstrand in [T]. We describe the elements for the case $\mathbb{F}=\mathbb{C}$, the other one is similar. First of all we take bipolar coordinates on $\mathbb{C}^n$ for $(x_1,y_1,\dots,x_n,y_n)$ we set $\tau=\sum_{j=1}^{p}(x_j^2+y_j^2)-\sum_{j=p+1}^{n}(x_j^2+y_j^2)$, $\rho=\sum_{j=1}^{n}(x_j^2+y_j^2)$, $u=(x_1,y_1,\dots,x_p,y_p)$, $v=(x_{p+1},y_{p+1},\dots,x_n,y_n)$. Hence $u=\left(\frac{\rho+\tau}{2}\right)^{\frac{1}{2}}\omega_u$, with $\omega_u\in S^{2p-1}$ and $v=\left(\frac{\rho-\tau}{2}\right)^{\frac{1}{2}}\omega_v$, with $\omega_v\in S^{2q-1}$. It is easy to see by changing variables that

$$\int_{\mathbb{C}^n} f(z)dz=\int_{-\infty}^{\infty}\int_{\rho>|\tau|}\int_{S^{2p-q}\times S^{2q-1}} f\left(\left(\frac{\rho+\tau}{2}\right)^{\frac{1}{2}}\omega_u,\left(\frac{\rho-\tau}{2}\right)^{\frac{1}{2}}\omega_v\right)d\omega_u d\omega_v\times$$
$$\times(\rho+\tau)^{p-1}(\rho-\tau)^{q-1}d\rho d\tau.$$

Then we define for $f\in\mathcal{S}(\mathbb{R}^{2n})$

$$Mf(\rho,\tau)=\int_{S^{2p-1}\times S^{2q-1}} f\left(\left(\frac{\rho+\tau}{2}\right)^{\frac{1}{2}}\omega_u,\left(\frac{\rho-\tau}{2}\right)^{\frac{1}{2}}\omega_v\right)d\omega_u d\omega_v,$$

and also

$$Nf(\tau)=\int_{|\tau|}^{\infty}Mf(\rho,\tau)(\rho+\tau)^{p-1}(\rho-\tau)^{q-1}d\rho.$$

Let us now define the space $\mathcal{H}_n$ of the functions $\varphi:\mathbb{R}\to\mathbb{C}$ such that $\varphi(\tau)=\varphi_1(\tau)+\tau^{n-1}\varphi_2(\tau)H(\tau)$, for $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R})$, where H denotes the Heaviside function. In [T] it is proved that $\mathcal{H}_n$ with a suitable topology is a Fréchet space, and following the same lines we can see that the linear maps $N:\mathcal{S}(\mathbb{R}^{2n}-\{0\})\to\mathcal{S}(\mathbb{R})$ and $N:\mathcal{S}(\mathbb{R}^{2n})\to\mathcal{H}$ are continuous and surjective. Let us consider now $\mu\in\mathcal{S}'(\mathbb{R}^{2n})^{U(p,q)}$, then it is easy to see that there exists a unique $T\in\mathcal{S}'(\mathbb{R})$ such that $< \mu, f >=<$

$T, Nf >$, for every $f \in \mathcal{S}(\mathbb{R}^{2n} - \{0\})$. Moreover, if $N' : \mathcal{H}' \to \mathcal{S}'(\mathbb{R}^{2n})$ is the adjoint map, by following again the arguments shown on [T] we can see that N' is a homeomorphism. Finally, for a function $f \in \mathcal{S}(N(p,q,\mathbb{C}))$ we write $Nf(\tau,t)$ for $N(f(.,t))(\tau)$.

The distributions $S_{\lambda,k}$ are defined as follows

$$S_{\lambda,k} = \sum_{\{m\in\mathbb{N}_o^n, B(m)=k\}} E_\lambda(h_m,h_m), \tag{1.5}$$

where $B(m) = \sum_{j=1}^{p} m_j - \sum_{j=p+1}^{n} m_j$, the set of functions $\{h_m\} \subset L^2(\mathbb{R}^n)$ is the Hermite basis and $E_\lambda(h,h')(z,t) =< \pi_\lambda(z,t)h, h' >$ are the matrix entries of the Schrödinger representation π_λ. Also, $S_{\lambda,k} = e^{-i\lambda t} \otimes F_{\lambda,k}$, where each $F_{\lambda,k} \in \mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$ is a tempered distribution defined in terms of the Laguerre polinomials L_k^m and the map N as follows: for $g \in \mathcal{S}(\mathbb{C}^n)$, $\lambda \neq 0$ and $k \in \mathbb{Z}$

$$\text{if } k \geq 0, < F_{\lambda,k}, g >=< (L^0_{k-q+n-1}H)^{(n-1)}, \tau \to \frac{2}{|\lambda|}e^{-\frac{\tau}{2}}Ng\left(\frac{2}{|\lambda|}\tau\right) >, \text{ and} \tag{1.6}$$

$$\text{if } k < 0, < F_{\lambda,k}, g >=< (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \to \frac{2}{|\lambda|}e^{-\frac{\tau}{2}}Ng\left(-\frac{2}{|\lambda|}\tau\right) > . \tag{1.7}$$

For the quaternionic case we consider the Schrödinger representation π_w as given in [R] (see also [K-R]):

$$\pi_w(\alpha, z) = \pi_{|w|}\left(\alpha, < z, \frac{w}{|w|} >\right), \tag{1.8}$$

where $\pi_{|w|}$ is the Schrödinger representation for the classical Heisenberg group $N(2p, 2q, \mathbb{C})$. We have analogously that the distributions $\varphi_{w,k}$ are defined by

$$\varphi_{w,k} = \sum_{m\in\mathbb{N}_0^{2n}:B(m)=k} E_w(h_m,h_m), \tag{1.9}$$

where $B(m) = \sum_{j=1}^{2p} m_j - \sum_{j=2p+1}^{2n} m_j$ and $E_w(h,h')(\alpha,z) =< \pi_w(\alpha,z)h, h' >$ are the matrix entries of the Schrödinger representation π_w. Moreover, we have that $\varphi_{w,k} = e^{i<w,z>} \otimes \theta_{w,k}$, where $\theta_{w,k}$ is a tempered distribution such that $\theta_{w,k} = N'T_{|w|,k}$. If we set $\lambda = |w|$, we have that $T_{|w|,k} = F_{\lambda,k}$ where we replace n, p, q by $2n, 2p, 2q$, respectively, in (1.6) and (1.7). Observe that if we define

$$\varphi_{\lambda,k}(\alpha, z) = \int_{S^2} e^{i<z,\lambda\xi>} d\xi \theta_{\lambda,k}(\alpha), \tag{1.10}$$

this distributions are $Spin(3) \otimes U(p,q,\mathbb{H})-$ invariant.

2 A fundamental solution for the operator $\mathcal{L}_\alpha$

We have that Φ_α defined as in (1.3) is a well defined tempered distribution, and a fundamental solution for $\mathcal{L}_\alpha$. We include the proof since a misprint in Lemma 1 of [M-R] is used in the proof Lemma 2.10 of [B-D-R].

We will consider $\alpha \in \mathbb{C}$ such that $2k + p - q \pm \alpha \neq 0$ for all $k \in \mathbb{Z}$.

Theorem 2.1. *Φ_α defined as in (1.3) is a well defined tempered distribution and it is a fundamental solution for the operator $\mathcal{L}_\alpha$.*

Proof. From (1.3) and (1.5) we can write

$$|<\Phi_\alpha, f>| \le \sum_{k\in\mathbb{Z}} \int_0^\infty \left(\left| \frac{<S_{-\lambda,k}, f>}{(2k+p-q+\alpha)} \right| + \left| \frac{<S_{\lambda,k}, f>}{(2k+p-q-\alpha)} \right| \right) |\lambda|^{n-1} d\lambda \le$$

$$\le \sum_{k\in\mathbb{Z}} \int_0^\infty \sum_{\substack{\beta\in\mathbb{N}_0^n \\ B(\beta)=k}} \left(\left| \frac{<E_{-\lambda}(h_\beta,h_\beta), f>}{(2k+p-q+\alpha)} \right| + \left| \frac{<E_{\lambda}(h_\beta,h_\beta), f>}{(2k+p-q-\alpha)} \right| \right) |\lambda|^{n-1} d\lambda.$$

From the known facts that $\sum_{k\in\mathbb{Z}} \sum_{\substack{\beta\in\mathbb{N}_0^n \\ B(\beta)=k}} p(\beta) = \sum_{k\ge 0} \binom{k+n-1}{n-1} p(k)$,
$|<E_\lambda(h_\beta,h_\beta), f>| = |<\pi_\lambda(f)h_\beta, h_\beta>| \le ||f||_{L^1(N(p,q,\mathbb{C}))}$ and that for $m \in \mathbb{N}$

$$\pi_\lambda(f)h_\beta = \frac{1}{(-1)^m|\lambda|^m(2B(\beta)+p-q+\alpha\ \mathrm{sgn}(\lambda))^m}\pi_\lambda(L^m f)h_\beta$$

we get that

$$|<\Phi_\alpha, f>| \le ||L^m f||_{L^1(N(p,q,\mathbb{C}))}\times$$
$$\times \sum_{k\ge 0} \int_0^\infty \binom{k+n-1}{k} \left(\frac{|\lambda|^{n-1-m}}{|2k+p-q+\alpha|^{m+1}} + \frac{|\lambda|^{n-1-m}}{|2k+p-q-\alpha|^{m+1}} \right) d\lambda.$$

Let us consider the first term, the second one is analogous. We split the integral between $|\lambda||2k+p-q+\alpha| \ge 1$ and $0 \le |\lambda||2k+p-q+\alpha| \le 1$.Thus,

$$\sum_{k\ge 0} \binom{k+n-1}{k} \int_{|\lambda||2k+p-q+\alpha|\ge 1} \frac{1}{|2k+p-q+\alpha|^{m+1}} |\lambda|^{n-1-m} d\lambda$$

is finite if we take $m > n$, and

$$\sum_{k\ge 0} \binom{k+n-1}{k} \int_{0\le|\lambda||2k+p-q+\alpha|\le 1} \frac{1}{|2k+p-q+\alpha|^{m+1}} |\lambda|^{n-1-m} d\lambda$$

is finite for $m < n$. From the above computations it also follows that Φ_α is a tempered distribution. Next we see that it is a fundamental solution by writing $L = L_0 + L_1$, which in coordinates are

$$L_0 = \frac{1}{4}\left(\sum_{j=1}^p (x_j^2+y_j^2) - \sum_{j=p+1}^n (x_j^2+y_j^2) \right) \frac{\partial^2}{\partial t^2} +$$
$$+ \sum_{j=1}^p \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) - \sum_{j=p+1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right),$$
$$L_1 = \frac{\partial}{\partial t} \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

Then, as L_0, L_1 and T commute with left translations and also $L_0(g^\vee) = (L_0 g)^\vee$, $L_1(g^\vee) = -(L_1 g)^\vee$ and $T(g^\vee) = -(Tg)^\vee$ we get that

$$(\mathcal{L}f * \Phi_\alpha)(z,t) =< \Phi_\alpha, (L_{(z,t)^{-1}}\mathcal{L}f)^\vee >=< \Phi_\alpha, (L_0 - i\alpha)(L_{(z,t)^{-1}}f)^\vee,$$

because $L_1\Phi_\alpha = 0$. Hence,

$$(\mathcal{L}_\alpha f * \Phi)(z,t) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{< S_{\lambda,k}, (L_0 - i\alpha T)(L_{(z,t)^{-1}} f)^\vee >}{-|\lambda|(2k+p-q-\alpha \text{ sgn } \lambda)} |\lambda|^{n-1} d\lambda =$$

$$= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{< (L_0 + i\alpha T)S_{\lambda,k}, (L_{(z,t)^{-1}} f)^\vee >}{-|\lambda|(2k+p-q-\alpha \text{ sgn } \lambda)} |\lambda|^{n-1} d\lambda =$$

$$= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} < S_{\lambda,k}, (L_{(z,t)^{-1}} f)^\vee > |\lambda|^{n-1} d\lambda = f(z,t),$$

because of the inversion formula. The other one, $f * \mathcal{L}_\alpha(f) = f$, is immediate. $\square$

Now we proceed with the computation of Φ_α. Given that the series (1.3) defining Φ_α converges absolutely, we can split the sum over $k \in \mathbb{Z}$ into the sums for $k \geq q$, for $k \leq -p$ and for $-p < k < q$. In the first case we change the summation index writing $k = k' + q$ and in the second as well, writing $k = k' - p$. So we get

$$< \Phi_\alpha, f >= (-1) \sum_{k'\geq 0} \frac{1}{2k'+n-\alpha} \int_0^\infty [< S_{\lambda,k'+q}, f > - < S_{\lambda,-k'-p}, f >] |\lambda|^{n-1} d\lambda +$$

$$+ (-1) \sum_{k'\geq 0} \frac{1}{2k'+n+\alpha} \int_0^\infty [< S_{-\lambda,k'+q}, f > - < S_{-\lambda,-k'-p}, f >] |\lambda|^{n-1} d\lambda +$$

$$+ (-1) \sum_{-p<k<q} \int_0^\infty \left(\frac{< S_{-\lambda,k}, f >}{2k+p-q+\alpha} + \frac{< S_{\lambda,k}, f >}{2k+p-q-\alpha} \right) |\lambda|^{n-1} d\lambda.$$

By Abel's Lemma and the Lebesgue dominated convergence theorem we can write $\Phi_\alpha = \Phi_1 + \Phi_2$ where

$$< \Phi_1, f >= \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} (-1) \sum_{k'\geq 0} \frac{r^{2k'+n-\alpha}}{2k'+n-\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \times \tag{2.1}$$

$$\times \left[< S_{\lambda,k'+q}, f > - < S_{\lambda,-k'-p}, f >\right] |\lambda|^{n-1} d\lambda +$$

$$+ \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} (-1) \sum_{k'\geq 0} \frac{r^{2k'+n+\alpha}}{2k'+n+\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \times$$

$$\times \left[< S_{-\lambda,k'+q}, f > - < S_{-\lambda,-k'-p}, f >\right] |\lambda|^{n-1} d\lambda,$$

$$< \Phi_2, f >= \lim_{\epsilon\to 0^+} (-1) \sum_{-p<k<q} \int_0^\infty e^{-\epsilon|\lambda|} \times \tag{2.2}$$

$$\times \left(\frac{< S_{-\lambda,k}, f >}{2k+p-q+\alpha} + \frac{< S_{\lambda,k}, f >}{2k+p-q-\alpha} \right) |\lambda|^{n-1} d\lambda.$$

Using that $S_{\lambda,k} = e^{-i\lambda t} \otimes F_{\lambda,k}$ and the computations from [G-S(2)], namely (2.6) to (2.9), we get that

$$< \Phi_1, f >= \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} (-1) \sum_{k\geq 0} \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \int_{-\infty}^\infty e^{-i\lambda t} \times$$
$$\times < (L^0_{k+n-1}H)^{(n-1)}, \frac{2}{|\lambda|} e^{-\frac{\tau}{2}} [Nf(\frac{2}{|\lambda|}\tau, t) - Nf(-\frac{2}{|\lambda|}\tau, t)] > dtd\lambda +$$
$$+ \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} (-1) \sum_{k\geq 0} \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \int_{-\infty}^\infty e^{i\lambda t} \times$$
$$\times < (L^0_{k+n-1}H)^{(n-1)}, \frac{2}{|\lambda|} e^{-\frac{\tau}{2}} [Nf(\frac{2}{|\lambda|}\tau, t) - Nf(-\frac{2}{|\lambda|}\tau, t)] > dtd\lambda,$$

and setting

$$b_{k,l} = \sum_{j=l}^{n-2} \binom{j}{l} \left(\frac{1}{2}\right)^{2-l} (-1)^{n-j} \binom{k+n-1}{n-j-2}, \tag{2.3}$$

we have that

$$< \Phi_1, f >= \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} \sum_{k\geq 0} \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \int_{-\infty}^\infty e^{-i\lambda t} \times$$
$$\times \left[(-1)^n \int_{-\infty}^\infty L_k^{n-1}\left(\frac{|\lambda|}{2}|s|\right) e^{-\frac{|\lambda|}{4}|s|} \operatorname{sgn}(s) Nf(s,t) ds + \right.$$
$$\left. -2 \sum_{\substack{l=0 \\ l \text{ odd}}}^{n-2} \left(\frac{2}{|\lambda|}\right)^{l+1} b_{k,l} \frac{\partial^l Nf}{\partial \tau^l}(0,t) \right] dtd\lambda +$$
$$+ \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} \sum_{k\geq 0} \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \int_{-\infty}^\infty e^{i\lambda t} \times$$
$$\times \left[(-1)^n \int_{-\infty}^\infty L_k^{n-1}\left(\frac{|\lambda|}{2}|s|\right) e^{-\frac{|\lambda|}{4}|s|} \operatorname{sgn}(s) Nf(s,t) ds + \right.$$
$$\left. -2 \sum_{\substack{l=0 \\ l \text{ odd}}}^{n-2} \left(\frac{2}{|\lambda|}\right)^{l+1} b_{k,l} \frac{\partial^l Nf}{\partial \tau^l}(0,t) \right] dtd\lambda.$$

Now we define

$$G_f(\tau, t) = Nf(\tau, t) - \sum_{j=0}^{n-2} \frac{\partial^j Nf}{\partial \tau^j}(0,t) \frac{\tau^j}{j!}, \tag{2.4}$$

and then we can split $\Phi_1 = \Phi_{11} + \Phi_{12}$ where

$$
\begin{aligned}
<\Phi_{11}, f> = \lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\geq 0}(-1)^n\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}|\lambda|^{n-1}\times \\
\times\int_{-\infty}^\infty L_k^{n-1}\left(\frac{|\lambda|}{2}|\tau|\right)e^{-\frac{|\lambda|}{4}|\tau|}\mathrm{sgn}(\tau)G_f(\tau,t)d\tau dt d\lambda + \\
+\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\geq 0}(-1)^n\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{i\lambda t}|\lambda|^{n-1}\times \\
\times\int_{-\infty}^\infty L_k^{n-1}\left(\frac{|\lambda|}{2}|\tau|\right)e^{-\frac{|\lambda|}{4}|\tau|}\mathrm{sgn}(\tau)G_f(\tau,t)d\tau dt d\lambda,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
<\Phi_{12}, f> = \lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\geq 0}\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}|\lambda|^{n-1}\times \\
\times 2\sum_{\substack{l=0\\ l\text{ odd}}}^{n-2}\left(\frac{2}{|\lambda|}\right)^{l+1}(a_{k,l}+b_{k,l})\frac{\partial^l Nf}{\partial\tau^l}(0,t)dt d\lambda + \\
+\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\geq 0}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{i\lambda t}|\lambda|^{n-1}\times \\
\times 2\sum_{\substack{l=0\\ l\text{ odd}}}^{n-2}\left(\frac{2}{|\lambda|}\right)^{l+1}(a_{k,l}+b_{k,l})\frac{\partial^l Nf}{\partial\tau^l}(0,t)dt d\lambda,
\end{aligned}
\tag{2.6}
$$

(2.7)

with

$$
a_{k,l} = (-1)^n\frac{1}{l!}\int_0^\infty L_k^{n-1}(s)e^{-\frac{s}{2}}s^l ds. \tag{2.8}
$$

We will show that Φ_{11} is well defined. We have proved that the series (1.3) defining Φ_α converges and as Φ_2 is a finite sum we will obtain that Φ_{12} is also well defined.

Proposition 2.2. *The following identities hold:*

(i)

$$
\begin{aligned}
\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}L_k^{n-1}\left(\frac{|\lambda|}{2}|\tau|\right)e^{-\frac{|\lambda|}{4}|\tau|}|\lambda|^{n-1}d\lambda = \\
= 4^n(n-1)!(-1)^n\binom{k+n-1}{k}\left(\frac{(|\tau|-4\epsilon-4it)^k}{(|\tau|+4\epsilon+4it)^{k+n}}\right).
\end{aligned}
$$

(ii)

$$\lim_{\epsilon\to 0^+}\int\limits_{\mathbb{R}^2}\left(\frac{(|\tau|-4it-4\epsilon)^k}{(|\tau|+4it+4\epsilon)^{k+n}}\right)\ sgn(\tau)G_f(\tau,t)d\tau dt=$$

$$=\int\limits_{\mathbb{R}^2}\frac{1}{(|\tau|-4it)^{\frac{n}{2}-\frac{\alpha}{2}}}\frac{1}{(|\tau|+4it)^{\frac{n}{2}+\frac{\alpha}{2}}}\left(\frac{|\tau|-4it}{\tau^2+16t^2}\right)^{2k+n-\alpha}\quad sgn(\tau)G_f(\tau,t)d\tau dt.$$

(iii)

$$\lim_{\epsilon\to 0^+}\int\limits_{\mathbb{R}^2}\left(\frac{(|\tau|+4it-4\epsilon)^k}{(|\tau|-4it+4\epsilon)^{k+n}}\right)\ sgn(\tau)G_f(\tau,t)d\tau dt=$$

$$=\int\limits_{\mathbb{R}^2}\frac{1}{(|\tau|-4it)^{\frac{n}{2}-\frac{\alpha}{2}}}\frac{1}{(|\tau|+4it)^{\frac{n}{2}+\frac{\alpha}{2}}}\left(\frac{|\tau|-4it}{\tau^2+16t^2}\right)^{2k+n+\alpha}\quad sgn(\tau)G_f(\tau,t)d\tau dt.$$

Proof. From (4.9) of [G-S(2)] we know that (i) follows from the generating identity for the Laguerre polynomials:

$$\sum_{k\geq 0}L_k^{n-1}(t)z^k=\frac{1}{(1-z)^n}e^{-\frac{zt}{1-z}}. \tag{2.9}$$

From Lemma 2.2 of [G-S(2)], which states that the function $\frac{G_f(\tau,t)}{(\tau^2+16t^2)^{\frac{n}{2}}}$ is integrable in $\mathbb{R}^2$ and the fact that $\left|\frac{1}{(|\tau|-4it)^{-\frac{\alpha}{2}}}\right|\left|\frac{1}{(|\tau|+4it)^{\frac{\alpha}{2}}}\right|=1$, it follows that the function $\frac{1}{(|\tau|-4it)^{\frac{n}{2}-\frac{\alpha}{2}}}\frac{1}{(|\tau|+4it)^{\frac{n}{2}+\frac{\alpha}{2}}}G_f(\tau,t)$ is integrable in $\mathbb{R}^2$. So we get (ii). For (iii) we just change $e^{-i\lambda t}$ by $e^{i\lambda t}$ and argue like for (ii). $\square$

Then, by Proposition 2.2,

$$\begin{aligned}<\Phi_{11},f>=\ &\beta_n\lim_{r\to 1^-}\sum_{k\geq 0}\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\alpha_k\times\\ &\times\int\limits_{\mathbb{R}^2}\left(\frac{|\tau|-4it}{\tau^2+16t^2}\right)^{2k+n-\alpha}\frac{\mathrm{sgn}(\tau)G_f(\tau,t)}{(|\tau|-4it)^{\frac{n}{2}-\frac{\alpha}{2}}(|\tau|+4it)^{\frac{n}{2}+\frac{\alpha}{2}}}d\tau dt+\\ &+\beta_n\lim_{r\to 1^-}\sum_{k\geq 0}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\alpha_k\times\\ &\int\limits_{\mathbb{R}^2}\left(\frac{|\tau|+4it}{\tau^2+16t^2}\right)^{2k+n+\alpha}\frac{\mathrm{sgn}(\tau)G_f(\tau,t)}{(|\tau|+4it)^{\frac{n}{2}+\frac{\alpha}{2}}(|\tau|-4it)^{\frac{n}{2}-\frac{\alpha}{2}}}d\tau dt,\end{aligned}$$

where $\beta_n=4^n(n-1)!(-1)^n$ y $\alpha_k=\binom{k+n-1}{k}(-1)^k$.

To study $<\Phi_{11},f>$ we split each integral over the left and right halfplanes and take polar

coordinates $\tau - 4it = \rho e^{i\theta}$ to obtain

$$
\begin{aligned}
< \Phi_{11}, f >= & \beta_n \lim_{r\to 1^-} \sum_{k\geq 0} \alpha_k \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \times \\
& \times \int_0^\infty \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(2k+n-\alpha)\theta} \frac{1}{4\rho^{n-1}} e^{i\alpha\theta} \text{ sgn } (\cos\theta) G_f(\rho\cos\theta, -\frac{\rho}{4}\sin\theta) d\theta + \right. \\
& \left. + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{e^{-i(2k+n-\alpha)\theta} e^{-i\alpha\theta}}{(-1)^n 4\rho^{n-1}} \text{ sgn } (\cos\theta) G_f(\rho\cos\theta, -\frac{\rho}{4}\sin\theta) d\theta \right] d\rho + \\
& + \beta_n \lim_{r\to 1^-} \sum_{k\geq 0} \alpha_k \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \times \\
& \times \int_0^\infty \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i(2k+n+\alpha)\theta} \frac{1}{4\rho^{n-1}} e^{i\alpha\theta} \text{ sgn } (\cos\theta) G_f(\rho\cos\theta, -\frac{\rho}{4}\sin\theta) d\theta + \right. \\
& \left. + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{e^{i(2k+n+\alpha)\theta} e^{-i\alpha\theta}}{(-1)^n 4\rho^{n-1}} \text{ sgn } (\cos\theta) G_f(\rho\cos\theta, -\frac{\rho}{4}\sin\theta) d\theta \right] d\rho
\end{aligned}
$$

Now we change variable in the second and fourth term according to $\theta \longleftrightarrow -\theta$. Then, in the fourth term we change variables again according to $\theta \longleftrightarrow \theta + 2\pi$. By proposition 2.2 we can interchange the integration order, so we can write

$$
\begin{aligned}
< \Phi_{11}, f >= & \beta_n \lim_{r\to 1^-} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha\theta} \times \\
& \times \left[\sum_{k\geq 0} \alpha_k \left(\frac{r^{2k+n-\alpha}}{2k+n-\alpha} e^{i(2k+n-\alpha)\theta} + \frac{r^{2k+n+\alpha}}{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta} \right) \right] \times \\
& \times \frac{1}{\rho^{n-1}} \text{ sgn } (\cos\theta) G_f(\rho\cos\theta, -\frac{\rho}{4}\sin\theta) d\theta d\rho \quad + \\
+ \quad & \frac{(-1)^n}{4} \beta_n \lim_{r\to 1^-} \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} e^{i\alpha\theta} \times \\
& \times \left[\sum_{k\geq 0} \alpha_k \left(\frac{r^{2k+n-\alpha}}{2k+n-\alpha} e^{i(2k+n-\alpha)\theta} + \frac{r^{2k+n+\alpha}}{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta} \right) \right] \times \\
& \times \frac{1}{\rho^{n-1}} \text{ sgn } (\cos\theta) G_f(\rho\cos\theta, \frac{\rho}{4}\sin\theta) d\theta d\rho.
\end{aligned}
$$

Let I denote the real interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Next let us consider the vector space

$$
\mathcal{X} = \left\{ g \in C^{n-2}(I) : g^{(j)}\left(\pm\frac{\pi}{2}\right) = 0,\ 0 \leq j \leq n-2,\ g^{(n-1)} \in L^\infty(I) \right\}.
$$

We identify each function $g \in \mathcal{X}$ with the function $\widetilde{g}$ on $S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$, defined equal to 0 outside supp(g) and we make no distinction between g and $\widetilde{g}$. Thus, if $g \in \mathcal{X}$ then $g \in C^{n-2}(S^1)$ with $g^{(n-1)} \in L^\infty(S^1)$. Observe that if $g \in \mathcal{X}$, then also $e^{i\alpha\theta} g \in \mathcal{X}$. The topology on $\mathcal{X}$ is given by $||g||_{\mathcal{X}} = \max_{0 \le j \le n-1} ||g^{(j)}||_\infty$.

For $k \in \mathbb{Z}$ we set $\alpha_k = \binom{k+n-1}{k}(-1)^k$. Now let us define

$$\Psi_{r,\alpha}(\theta) = \sum_{k \ge 0} \alpha_k \left(\frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right), \tag{2.10}$$

$$< \Psi_\alpha, g > = < \sum_{k \ge 0} \alpha_k \left(\frac{e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right), g >, \tag{2.11}$$

and let us see that $\Psi_\alpha \in \mathcal{X}'$, the dual space of $\mathcal{X}$. Indeed,

$$| < \Psi_\alpha, g > | \quad \le \quad |e^{i\alpha\theta}| \sum_{k \ge 0} \binom{k+n-1}{k} \left(\frac{| < e^{i(2k+n)\theta}, g > |}{|2k+n-\alpha|} + |\frac{< e^{-i(2k+n)\theta}, g > |}{|2k+n+\alpha|} \right). \tag{2.12}$$

If $\widehat{g}(n) = < g, e^{in\theta} >$ denotes the $n-$th Fourier coefficient of g, then we have that

$$\begin{aligned} | < \Psi_\alpha, g > | &\le c \sum_{k \ge 0} \frac{k^{n-1}}{|2k+n|^{n-1}} \left(\frac{|\widehat{g^{(n-1)}}(2k+n)|}{|2k+n-\alpha|} + \frac{|\widehat{g^{(n-1)}}(-2k-n)|}{|2k+n+\alpha|} \right) \le \\ &\le c \sum_{k \ge 0} \frac{1}{k} |\widehat{g^{(n-1)}}(2k+n)| + \frac{1}{k} |\widehat{g^{(n-1)}}(-2k-n)| \le \\ &\le c \left(\sum_{k \ge 0} \frac{1}{k^2} \right)^{\frac{1}{2}} ||\widehat{g^{(n-1)}}||_{L^2}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality. Observe that the constants c are not the same on each expression.

By Abel's Lemma, $\lim_{r \to 1^-} \Psi_{r,\alpha} = \Psi_\alpha$ in $\mathcal{X}'$ that is, with respect to the weak convergence topology. Similarly, if J denotes the real interval $\left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$, we define the space,

$$\mathcal{Y} = \left\{ g \in C^{n-2}(J) : g^{(j)}\left(\frac{\pi}{2}\right) = g^{(j)}\left(\frac{3}{2}\pi\right) = 0,\, 0 \le j \le n-2,\, g^{(n-1)} \in L^\infty(J) \right\},$$

and we obtain that Ψ_α is well defined in $\mathcal{Y}'$ and $\lim_{r \to 1^-} \Psi_{r,\alpha} = \Psi_\alpha$ in $\mathcal{Y}'$.

Our aim now is to compute Ψ_α.

From Proposition 3.7 of [G-S(2)] we know that if $\Theta \in \mathcal{D}'(S^1)$ is defined by

$$\Theta(\theta) = i \sum_{k \ge 0} \binom{k+n-1}{k} (-1)^k e^{i(2k+n)\theta}, \tag{2.13}$$

then for even n we have that

$$\mathfrak{Re}\Theta(\theta) = \frac{d}{d\theta} Q_{n-2}\left(\frac{d}{d\theta}\right)(\delta_{\frac{\pi}{2}} + \delta_{-\frac{\pi}{2}}) = \sum_{j=0}^{n-2} c_j \left(\delta_{\frac{\pi}{2}}^{(j+1)} + \delta_{-\frac{\pi}{2}}^{(j+1)} \right), \tag{2.14}$$

where Q_{n-2} is a polynomial of degree $n-2$; and for odd n we have that

$$\mathfrak{Re}\Theta(\theta) = d_0 \frac{d}{d\theta} \widetilde{H} + \frac{d}{d\theta} Q_{n-2}\left(\frac{d}{d\theta}\right)(\delta_{\frac{\pi}{2}} - \delta_{-\frac{\pi}{2}}) = d_0(\delta_{-\frac{\pi}{2}} - \delta_{\frac{\pi}{2}}) + \sum_{j=0}^{n-2} c_j (\delta_{\frac{\pi}{2}}^{(j+1)} - \delta_{-\frac{\pi}{2}}^{(j+1)}), \tag{2.15}$$

where Q_{n-2} is a polynomial of degree $n-2$, and $\widetilde{H}(\theta) = H(\cos\theta)$.

Let us recall the generating identity for the Laguerre polynomials (2.9), and take $t = 0$ and $z = -r^2 e^{2i\theta}$. We get

$$\sum_{k\geq 0} \binom{k+n-1}{k} (-1)^k r^{2k+n} e^{i(2k+n)\theta} = \left(\frac{re^{i\theta}}{1+r^2 e^{2i\theta}} \right)^n. \tag{2.16}$$

We also need a couple of results:

Lemma 2.3. *For a fixed $r > 1$ the functions $\alpha \to \Psi_{r,\alpha}(0)$ and $\alpha \to \lim\limits_{r\to 1^-} \Psi_{r,\alpha}(0)$ are analytic on $\Omega = \mathbb{C}\backslash F$, where $F = \{2k+n : k \in \mathbb{Z}\}$.*

Proof. Let K be a compact set, $K \subset \Omega$. It is easy to see that for fixed r the series (2.10) converges uniformly, since

$$|\Psi_{r,\alpha}(0)| \leq \max_{\alpha\in K} |r^\alpha| \left(\frac{r}{1+r^2} \right)^n d(K,F).$$

Also, for $\alpha \in \Omega$ there exists $\lim\limits_{r\to 1^-} \Psi_{r,\alpha}(0)$. Indeed, if $0 \leq r_1 < r < r_2 < 1$, from the Mean Value Theorem we have that for some $\xi \in (r_1, r_2)$,

$$\Psi_{r_1,\alpha}(0) - \Psi_{r_2,\alpha}(0) = \frac{d}{dr}\Psi_{\xi,\alpha}(0)(r_2 - r_1) = (\xi^{-\alpha-1} + \xi^{\alpha-1}) \sum_{k\geq 0} \alpha_k \xi^{2k+n}(r_2 - r_1) =$$
$$= (\xi^{-\alpha-1} + \xi^{\alpha-1}) \left(\frac{\xi}{1+\xi^2} \right)^n (r_2 - r_1),$$

where the last equality holds from (2.16). Hence

$$|\Psi_{r_1,\alpha}(0) - \Psi_{r_2,\alpha}(0)| \leq c(\xi)|r_2 - r_1|,$$

where $c(\xi)$ is a constant which depends on ξ. Moreover, for $\alpha \in K$ and $\xi \in \left[\frac{1}{2}, 1\right]$, $\xi^{n-\alpha-1} + \xi^{n+\alpha-1}$ is bounded in $K \times \left[\frac{1}{2}, 1\right]$, so the convergence is uniform, hence $\alpha \to \lim\limits_{r\to 1^-} \Psi_{r,\alpha}(0)$ is an analytic function. ☐

Lemma 2.4. *Let $0 < \delta < \frac{\pi}{4}$. For $0 < r < 1$ and $0 \leq |\theta| < \delta$ we have that*

$$|\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)| \leq \left(\max_{0\leq|\theta|<\delta} e^{|\mathfrak{Im}\alpha||\theta|} \right) \left(a|r^{-\alpha} - r^\alpha| + b|r^\alpha|(1-r) \right) |\theta|,$$

with a, b positive constants. Also for $0 \leq |\theta - \pi| < \delta < \frac{\pi}{4}$,

$$|\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(\pi)| \leq \left(\max_{0\leq|\theta-\pi|<\delta} e^{|\mathfrak{Im}\alpha||\theta|} \right) \left(a|r^{-\alpha} - r^\alpha| + b|r^\alpha|(1-r) \right) |\theta - \pi|,$$

with a, b positive constants.

Proof. We will estimate $|\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)|$ for $0 < |\theta| < \delta < \frac{\pi}{4}$, the other case is similar. We have

$$\frac{d}{d\theta}\Psi_{r,\alpha}(\theta) = ie^{-i\alpha\theta} \sum_{k\geq 0} \alpha_k r^{2k+n} \left((r^{-\alpha} - r^\alpha) e^{i(2k+n)\theta} + (e^{i(2k+n)\theta} - e^{-i(2k+n)\theta}) r^\alpha \right) =$$
$$= ie^{-i\alpha\theta} \left((r^{-\alpha} - r^\alpha) \left(\frac{re^{i\theta}}{1+r^2 e^{2i\theta}} \right)^n + 2ir^\alpha \mathfrak{Im} \left(\frac{re^{i\theta}}{1+r^2 e^{i2\theta}} \right)^n \right),$$

because of (2.16). We have that

$$\left|\frac{d}{d\theta}\Psi_{r,\alpha}(\theta)\right| \leq e^{|\mathfrak{Im}\alpha||\theta|}\left(|r^{-\alpha}-r^{\alpha}|\left|\left(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\right)^n\right|+2|r^{\alpha}|\left|\mathfrak{Im}\left(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\right)^n\right|\right). \tag{2.17}$$

From Proposition 3.1 of [G-S(2)] we know that $\left|\mathfrak{Im}\left(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\right)^n\right| \longrightarrow 0$ as $r \to 1^-$, uniformly for $|\theta| < \frac{\pi}{4}$, $|\theta-\pi| < \frac{\pi}{4}$. Also, $\left|\left(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\right)^n\right| \leq c$, for a constant c. Then $\left|\frac{d}{d\theta}\Psi_{r,\alpha}(\theta)\right| \longrightarrow 0$ uniformly on $|\theta| < \frac{\pi}{4}$ as $r \to 1^-$, and we get the desired inequality by applying the Mean Value Theorem around 0. □

Now we can state the following

Proposition 2.5. *For $f \in \mathcal{X}$ we have that $< \Psi_\alpha, f >= C_\alpha < 1, f >$, where $C_\alpha = \frac{\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{n-\alpha}{2})}{(n-1)!}$; and for $f \in \mathcal{Y}$ we have that $< \Psi_\alpha, f >= \widetilde{C_\alpha} < 1, f >$, where $\widetilde{C_\alpha} = (-1)^n e^{-i\alpha\pi}C_\alpha$.*

Proof. First we consider $f \in \mathcal{X}$ such that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt = 0$ and we define $F(\theta) = \int_{-\frac{\pi}{2}}^{\theta} f(t)dt$. It is easy to see that $F \in \mathcal{X}$ and $F' = f$. Because of the integration by parts formula we have that

$$< \Psi_\alpha, f >=< \Psi_\alpha, F' >= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{k\geq 0}\alpha_k\left(\frac{e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha}+\frac{e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha}\right)F'(\theta)d\theta =$$
$$=-< \Theta, e^{-i\alpha\theta}F > - < \overline{\Theta}, e^{-i\alpha\theta}F >,$$

where $\overline{\Theta} = \sum_{k\geq 0}\binom{k+n-1}{k}(-1)^k e^{-i(2k+n)\theta}$. So if n is even, from (2.14) we get that

$$< \Psi_\alpha, f >= -\sum_{j=0}^{n-2} c_j < \delta^{(j+1)}_{\frac{\pi}{2}} + \delta^{(j+1)}_{\frac{-\pi}{2}}, e^{-i\alpha\theta}F > -\sum_{j=0}^{n-2}\overline{c_j} < \overline{\delta^{(j+1)}_{\frac{\pi}{2}}} + \overline{\delta^{(j+1)}_{\frac{-\pi}{2}}}, e^{-i\alpha\theta}F >,$$

and because $< \delta^{(j+1)}_{\pm\frac{\pi}{2}}, e^{-i\alpha\theta}F >= 0$ we conclude that $< \Psi_\alpha, f >= 0$. If n is odd we use (2.15) to conclude that $< \Psi_\alpha, f >= 0$.

For the general case of any $f \in \mathcal{X}$ we consider $h \in \mathcal{X}$ such that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(t)dt = 1$ and define $g(\theta) = f(\theta) - \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt\right)h(\theta)$. So we can apply the above result to g and get that $< \Psi_\alpha, g >= 0$. Then $< \Psi_\alpha, f >=< \Psi_\alpha, g > + < \Psi_\alpha, h >< 1, f >=< \Psi_\alpha, h >< 1, f >$. Let $C_\alpha =< \Psi_\alpha, h >$.

In order to compute C_α, consider $g \in \mathcal{X}$ such that $\operatorname{supp}(g) \subset (-\frac{\pi}{4}, \frac{\pi}{4})$, $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} g(t)dt = 1$ and $g \geq 0$ we have that

$$< e^{i\alpha\theta}\Psi_\alpha, g >= C_\alpha \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha\theta}g(\theta)d\theta,$$

and also that

$$< e^{i\alpha\theta}\Psi_\alpha, g >= \lim_{r\to 1^-}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\Psi_{r,\alpha}(\theta)-\Psi_{r,\alpha}(0))e^{i\alpha\theta}g(\theta)d\theta + \Psi_{r,\alpha}(0)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha\theta}g(\theta)d\theta\right).$$

From lemmas 1 and 2 we get that

$$C_\alpha = \lim_{r\to 1^-} \Psi_{r,\alpha}(0)$$

and also that C_α is an analytic function of α. Given that $\Psi_{0,\alpha}(0) = 0$ we can write

$$C_\alpha = \lim_{r\to 1^-} \Psi_{r,\alpha}(0) = \Psi_{1,\alpha}(0) - \Psi_{0,\alpha}(0) = \int_0^1 w'_\alpha(s)ds,$$

where

$$w_\alpha(r) = \Psi_{r,\alpha}(0) = r^{-\alpha} \sum_{k\geq 0} \alpha_k \frac{r^{2k+n}}{2k+n-\alpha} + r^\alpha \sum_{k\geq 0} \alpha_k \frac{r^{2k+n}}{2k+n+\alpha}.$$

Applying (2.9) with $\theta = 0$ we obtain $w'_\alpha(r) = (r^{-\alpha-1} + r^{\alpha-1}) \sum_{k\geq 0} \alpha_k r^{2k+n} = (r^{-\alpha-1} + r^{\alpha-1}) \left(\frac{r}{1+r^2}\right)^n$, and we can solve the integral for $\mathfrak{Re}(n+\alpha) > 0$, $\mathfrak{Re}(n-\alpha) > 0$, getting

$$C_\alpha = B\left(\frac{n+\alpha}{2}, \frac{n-\alpha}{2}\right) = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{(n-1)!}, \tag{2.18}$$

where B is the *Beta function* and Γ is the *Gamma function*. By lemma 2.3, (2.18) holds for the other range of α, by analytic continuation. In a completely analogous way we get that $\widetilde{C_\alpha} = (-1)^n e^{-i\alpha\pi} C_\alpha$. □

Let us now define

$$K_{1f}(\rho,\theta) = \frac{1}{\rho^{n-1}} \operatorname{sgn}(\cos\theta) G_f(\rho\cos\theta, -\frac{\rho}{4}\sin\theta), \tag{2.19}$$

for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $0 < \rho < \infty$, where G_f is the function defined in (2.4); and

$$K_{2f}(\rho,\theta) = \frac{1}{\rho^{n-1}} \operatorname{sgn}(\cos\theta) G_f(\rho\cos\theta, \frac{\rho}{4}\sin\theta), \tag{2.20}$$

for $\theta \in \left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$, $0 < \rho < \infty$.

It is easy to check that $K_{1f}(\rho, .) \in \mathcal{X}$. Recall that we changed variables according to $\tau - 4it = \rho e^{i\theta}$. Since $Nf \in \mathcal{H}_n$, there exists a positive constant c such that

$$\sup_{\tau\neq 0, t\in\mathbb{R}} |(\tau^2 + 16t^2) Nf(\tau, t)| \leq c,$$

that is

$$\left|Nf(\rho\cos\theta, -\frac{\rho}{4}\sin\theta)\right| \leq \frac{c}{\rho^2}.$$

Also since $Nf(0, .) \in \mathcal{S}(\mathbb{R})$, there exists a positive constant c_N such that for $t \in \mathbb{R}$,

$$\left| t^N \sum_{j=0}^{n-2} \frac{\partial^j}{\partial\tau^j} Nf(0,t) \frac{\tau^j}{j!} \right| \leq c_N |\tau|^{n-2}.$$

Thus, for $N \in \mathbb{N}$ there exists c_N such that

$$|K_{1f}(\rho,\theta)| \leq \frac{a}{\rho^{n+1}} + \frac{b}{\rho^{N+1}} \frac{|\cos\theta|^{n-2}}{|\sin\theta|^N}. \tag{2.21}$$

Analogous observations are also true for K_{2f}.

Proposition 2.6. *Let C_α and $\widetilde{C_\alpha}$ be the constans obtained in (2.18). Let K_{1f} and K_{2f} be as defined in (2.19) and (2.20) and $\alpha_k = \binom{k+n-1}{k}(-1)^k$. Then*

$$\lim_{r\to 1^-} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha\theta} \sum_{k\geq 0} \alpha_k \left(\frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right) K_{1f}(\rho,\theta) d\theta d\rho =$$

$$= 4^{n-1}(n-1)! C_\alpha \int_{\mathbb{R}} \int_{\tau>0} \frac{1}{(\tau-4it)^{\frac{n-\alpha}{2}}} \frac{1}{(\tau+4it)^{\frac{n+\alpha}{2}}} sgn(\tau) G_f(\tau,t) d\tau dt,$$

and

$$\lim_{r\to 1^-} \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} e^{i\alpha\theta} \sum_{k\geq 0} \alpha_k \left(\frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right) K_{2f}(\rho,\theta) d\theta d\rho =$$

$$= 4^{n-1}(n-1)! \widetilde{C_\alpha} \int_{\mathbb{R}} \int_{\tau<0} \frac{1}{(\tau-4it)^{\frac{n-\alpha}{2}}} \frac{1}{(\tau+4it)^{\frac{n+\alpha}{2}}} sgn(\tau) G_f(\tau,t) d\tau dt.$$

Proof. The proof follows the same lines that Proposition 4.2 of [G-S(2)]. We only sketch it for completeness.

Taking polar coordinates $\tau - 4it = \rho e^{i\theta}$ we only need to show that

$$\lim_{r\to 1^-} \int_0^\infty < \Psi_{r,\alpha}, e^{i\alpha\theta} K_{1f}(\rho,\theta) > d\rho = \int_0^\infty < C_\alpha, e^{i\alpha\theta} K_{1f}(\rho,\theta) > d\rho. \tag{2.22}$$

In order to do this we split the integral for $0 < \rho < 1$ and $1 < \rho < \infty$.

We consider first the case $1 < \rho < \infty$. For $|\theta| \leq \delta < \frac{\pi}{4}$, set $I = \int_1^\infty \int_{|\theta|<\delta} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)) K_{1f}(\rho,\theta) d\theta d\rho$ and $II = \int_1^\infty \int_{|\theta|<\delta} e^{i\alpha\theta}(\Psi_{r,\alpha}(0) - C_\alpha) K_{1f}(\rho,\theta) d\theta d\rho$. We bound I close to 0 by applying Lemma 2.4 and taking $N = 1$ in (2.21). For II we just take $N = \frac{1}{2}$ in (2.21). To analize the case $\delta \leq |\theta| \leq \frac{\pi}{2}$, we observe that the function $K_{1f}^*(\theta) = \int_1^\infty K_{1f}(\rho,\theta) d\rho$ defined for $\theta \in [-\frac{\pi}{2}, -\delta] \cup [\delta, \frac{\pi}{2}]$ can be extended to an element of $\mathcal{X}$ that we still denote by K_{1f}^*. Then

$$\int_1^\infty \int_{\delta<|\theta|<\frac{\pi}{2}} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta) - C_\alpha) K_{1f}(\rho,\theta) d\theta d\rho =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta) - C_\alpha) K_{1f}^*(\theta) d\theta - \int_{|\theta|<\delta} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta) - C_\alpha) K_{1f}^*(\theta) d\theta.$$

The first term converges to zero as $r \to 1^-$ since $\Psi_{r,\alpha} \to C_\alpha$ as $r \to 1^-$ in $\mathcal{X}'$. For the second term we argue as above.

Finally, for the case $0 < \rho < 1$ we apply the same arguments to the function $K_{1f}^{**}(\theta) = \int_0^1 K_{1f}(\rho,\theta) d\rho$. □

Corollary 2.7. *$<\Phi_{11}, f>$ is well defined for $f \in \mathcal{S}(\mathbb{H}_n)$, and*

$$< \Phi_{11}, f >= 4^{n-1}(n-1)! C_\alpha \int\limits_{\mathbb{R}} \int\limits_{\tau>0} \frac{1}{(\tau - 4it)^{\frac{n-\alpha}{2}}} \frac{1}{(\tau + 4it)^{\frac{n+\alpha}{2}}} sgn(\tau) G_f(\tau, t) d\tau dt +$$
$$+ 4^{n-1}(n-1)! \widetilde{C_\alpha} \int\limits_{\mathbb{R}} \int\limits_{\tau<0} \frac{1}{(\tau - 4it)^{\frac{n-\alpha}{2}}} \frac{1}{(\tau + 4it)^{\frac{n+\alpha}{2}}} sgn(\tau) G_f(\tau, t) d\tau dt.$$

From the corollary we also get that $< \Phi_{12}, f >$ is well defined. In order to explicitly compute it we define for $0 \le l \le n-2$, $\epsilon > 0$ and $f \in \mathcal{S}(\mathbb{H}_n)$

$$d^-_{\epsilon,l,f} = \int_0^\infty \int_{-\infty}^\infty e^{-\epsilon|\lambda|} e^{-i\lambda t} |\lambda|^{n-l-2} \frac{\partial^l}{\partial \tau^l} Nf(0,t) dt d\lambda, \quad \text{and} \tag{2.23}$$

$$d^+_{\epsilon,l,f} = \int_0^\infty \int_{-\infty}^\infty e^{-\epsilon|\lambda|} e^{i\lambda t} |\lambda|^{n-l-2} \frac{\partial^l}{\partial \tau^l} Nf(0,t) dt d\lambda. \tag{2.24}$$

Then we can write (2.7) as

$$< \Phi_{12}, f >= \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \ge 0} \sum_{\substack{l=0 \\ l \text{ odd}}}^{n-2} 2^{l+2}(a_{kl} + b_{kl}) \left[\frac{r^{2k+n-\alpha}}{2k+n-\alpha} d^-_{\epsilon,l,f} + \frac{r^{2k+n+\alpha}}{2k+n+\alpha} d^+_{\epsilon,l,f} \right].$$

From Lemma 4.4 in [G-S(2)] we deduce that

$$a_{kl} + b_{kl} = (-1)^k \sum_{j=1}^{l+1} \frac{1}{2^{n-l-j-1}} \binom{n-j-1}{l-j+1} \binom{j+k-1}{k}.$$

Also we have the following

Lemma 2.8. *If $0 \le l \le n-2$, $\epsilon > 0$ and $f \in \mathcal{S}(\mathbb{H}_n)$, then*

$$\lim_{\epsilon \to 0^+} d^-_{\epsilon,l,f} = \frac{1}{i^{n-l-2}} < \frac{\pi}{2}\delta - i(vp)\left(\frac{1}{\lambda}\right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} Nf(0,.) > \qquad \textit{and}$$

$$\lim_{\epsilon \to 0^+} d^+_{\epsilon,l,f} = i^{n-l-2} < \frac{\pi}{2}\delta + i(vp)\left(\frac{1}{\lambda}\right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} Nf(0,.) > .$$

Proof. Let us consider $g(\lambda) = e^{-\epsilon|\lambda|}|\lambda|^{n-l-2}$ and $h(t) = \frac{\partial^l}{\partial \tau^l} Nf(0,t)$, and observe that $\int_{-\infty}^{\infty} e^{-i\lambda t} h(t) dt = \hat{h}(\lambda)$. Then just by using properties of the Fourier transform we get that

$$d^-_{\epsilon,l,f} = \int_0^\infty \int_{-\infty}^\infty g(\lambda) e^{-i\lambda t} h(t) dt d\lambda = \int_0^\infty g(\lambda) \hat{h}(\lambda) d\lambda = \frac{1}{i^{n-l-2}} \int_{-\infty}^\infty \frac{1}{\epsilon + i\lambda} h^{(n-l-2)}(\lambda) d\lambda.$$

For each $\epsilon > 0$, $\frac{1}{\epsilon+i\lambda}$ is a distribution such that there exists $\lim\limits_{\epsilon \to 0^+} \frac{1}{\epsilon+i\lambda}$ in $\mathcal{S}'(\mathbb{R})$. Moreover, it is easy to check that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon + i\lambda} = \frac{\pi}{2}\delta - i(vp)\left(\frac{1}{\lambda}\right).$$

Thus the desired equality follows. For $d^+_{\epsilon,l,f}$ we need to change variable according to $\lambda \longleftrightarrow -\lambda$ after considering the Fourier transform of h. ☐

We define for $j \in \mathbb{N}$, $0 < j < n-1$, the functions of r, with $0 \leq r < 1$, by

$$w_j^-(r) = \sum_{k \geq 0} (-1)^k \binom{j+k-1}{k} \frac{r^{2k+n-\alpha}}{2k+n-\alpha},$$

$$w_j^+(r) = \sum_{k \geq 0} (-1)^k \binom{j+k-1}{k} \frac{r^{2k+n+\alpha}}{2k+n+\alpha}.$$

We can see, in a complete analogous way as the computations made for C_α and $\widetilde{C_\alpha}$, that this functions are well defined and that

$$\begin{array}{rcl} c_j^- &:=& \lim\limits_{r \to 1^-} w_j^-(r) = \frac{1}{2} B_{\frac{1}{2}} \left(\frac{n-\alpha}{2}, j - \frac{n-\alpha}{2} \right), \text{ and} \\ c_j^+ &:=& \lim\limits_{r \to 1^-} w_j^+(r) = \frac{1}{2} B_{\frac{1}{2}} \left(\frac{n+\alpha}{2}, j - \frac{n+\alpha}{2} \right), \end{array} \tag{2.25}$$

where $B_{\frac{1}{2}}$ is another special function called the *incomplete Beta function.*

We now plug all of this definitions and results together to finally obtain an expression for Φ_{12}:

$$\begin{aligned} < \Phi_{12}, f >= & \sum_{\substack{l=0 \\ l \text{ odd}}}^{n-2} \sum_{j=1}^{l+1} 2^{2l-n+j+3} \binom{n-j-1}{l-j+1} \left[\left(\frac{1}{i^{n-l-2}} c_j^- + i^{n-l-2} c_j^+ \right) \frac{\pi}{2} \right] \times \\ & \times < \delta, \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N f(0,.) > + \\ & + (-1) \sum_{\substack{l=0 \\ l \text{ odd}}}^{n-2} \sum_{j=1}^{l+1} 2^{2l-n+j+3} \binom{n-j-1}{l-j+1} \left(\frac{1}{i^{n-l+1}} c_j^- + i^{n-l+1} c_j^+ \right) \times \\ & \times < (vp) \left(\frac{1}{\lambda} \right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N f(0,.) > . \end{aligned}$$

All we need to do now is to use again lemma 2.8 to get an expression for Φ_2. Thus, we have proved the following

Theorem 2.9. *Let C_α and $\widetilde{C_\alpha}$ be the constans defined as in (2.18), and let c_j^- and c_j^+ the constants*

defined as in (2.25). Then

$$< \Phi, f >= 4^{n-1}(n-1)!C_\alpha \int\limits_{\mathbb{R}} \int\limits_{\tau>0} \frac{1}{(\tau-4it)^{\frac{n-\alpha}{2}}} \frac{1}{(\tau+4it)^{\frac{n+\alpha}{2}}} sgn(\tau) G_f(\tau,t) d\tau dt +$$

$$+ 4^{n-1}(n-1)!\widetilde{C_\alpha} \int\limits_{\mathbb{R}} \int\limits_{\tau<0} \frac{1}{(\tau-4it)^{\frac{n-\alpha}{2}}} \frac{1}{(\tau+4it)^{\frac{n+\alpha}{2}}} sgn(\tau) G_f(\tau,t) d\tau dt +$$

$$+ \sum_{\substack{l=0 \\ l \ odd}}^{n-2} \sum_{j=1}^{l+1} 2^{2l-n+j+3} \binom{n-j-1}{l-j+1} \left[\left(\frac{1}{i^{n-l-2}} c_j^- + i^{n-l-2} c_j^+ \right) \frac{\pi}{2} \right] \times$$

$$\times < \delta, \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N f(0,.) > +$$

$$+ (-1) \sum_{\substack{l=0 \\ l \ odd}}^{n-2} \sum_{j=1}^{l+1} 2^{2l-n+j+3} \binom{n-j-1}{l-j+1} \left(\frac{1}{i^{n-l+1}} c_j^- + i^{n-l+1} c_j^+ \right) \times$$

$$\times < (vp) \left(\frac{1}{\lambda} \right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N f(0,.) > +$$

$$+ \sum_{k=0}^{n-2} \sum_{l=0}^{n-2} c_{kl} \left[\frac{1}{n-2k+\alpha-2} i^{n-l-2} < \frac{\pi}{2}\delta + i(vp)\left(\frac{1}{\lambda}\right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N f(0,.) > + \right.$$

$$\left. + \frac{1}{n-2k-\alpha-2} \frac{1}{i^{n-l-2}} < \frac{\pi}{2}\delta - i(vp)\left(\frac{1}{\lambda}\right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N f(0,.) > \right],$$

where $c_{kl} = \sum\limits_{\substack{1 \le j \le n-2 \\ j \ge n-k-2}} 2^{2l-n-j} (-1)^{n-j} \binom{j}{l} \binom{k-l+1}{n-j-2}$.

3 A fundamental solution for L

Like in the classical case, we have that the distribution Φ defined in (1.4) is a well defined tempered distribution and it is a relative fundamental solution for the operator L. The proof is identical to the one of theorem 2.1.

We will compute the fundamental solution Φ by means of the Radon transform and the fundamental solution of the operator L in the classical case $\mathbb{H}_{2n}$.

Let $F \in \mathcal{S}(\mathbb{R}^3)$. We assign to F a function $\mathcal{R}F : \mathbb{R} \times S^2 \to \mathbb{R}$ defined by

$$\mathcal{R}F(t,\xi) = \int\limits_{\mathbb{R}^2} F(t\xi + u_1 e_1 + u_2 e_2) du_1 du_2,$$

where $\{\xi, e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^3$. It is easy to see that this definition does not depend on the choice of the basis. In order to recover F from $\mathcal{R}F$, we consider the space $\mathcal{S}(\mathbb{R} \times S^2)$ of the continuous functions $G : \mathbb{R} \times S^2 \to \mathbb{R}$ that are infinitely differentiable in t and satisfy for every $m, k \in \mathbb{N}_0$ that

$$\sup_{t \in \mathbb{R}, \xi \in S^2} \left| t^m \frac{\partial^k}{\partial t^k} G(t,\xi) \right| < \infty.$$

Now for $G \in \mathcal{S}(\mathbb{R} \times S^2)$ we define a function $\mathcal{R}^* G : \mathbb{R}^3 \to \mathbb{R}$ by

$$\mathcal{R}^* G(z) = \int\limits_{S^2} G(< z, \xi >, \xi) d\xi.$$

Both assignments are well defined, and moreover $\mathcal{R}:\mathcal{S}(\mathbb{R}^3)\to\mathcal{S}(\mathbb{R}\times S^2)$ is the *Radon transform*, $\mathcal{R}^*:\mathcal{S}(\mathbb{R}\times S^2)\to\mathcal{S}(\mathbb{R}^3)$ is the *dual Radon transform* and they satisfy for every $F\in\mathcal{S}(\mathbb{R}^3)$

$$-2\pi F=\triangle\mathcal{R}^*\mathcal{R}F, \tag{3.1}$$

where $\triangle=\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\frac{\partial^2}{\partial z_3^2}$ is the $\mathbb{R}^3-$Laplacian.

Now let us consider the function ϕ defined for a fixed τ, $\tau\neq 0$ by

$$\phi(\tau,z)=\frac{16n}{\pi}\frac{4^{2n}(2n-1)!c_0}{(\tau^2+16|z|^2)^{n+1}},$$

where $c_0=-\int\limits_0^1\sigma^{2n-1}(1+\sigma^2)^{2n}d\sigma$. The function $\phi(\tau,.)$ is not a Schwartz function on $\mathbb{R}^3$, but we have that $(1+\triangle)^k\phi(\tau,.)\in L^1(\mathbb{R}^3)$, hence $(1+|\xi|^2)^k\widehat{\phi(\tau,.)}(\xi)\in L^\infty(\mathbb{R}^3)$. With these properties the inversion formula for the Radon transform (3.1) still holds. The proof follows straightforward from theorem 5.4 of [S-Sh]. Let us now compute the Radon transform of the function ϕ.

$$\begin{aligned}\mathcal{R}\phi(\tau,t,\xi)&=\int\limits_{\mathbb{R}^2}\frac{16n}{\pi}\frac{4^{2n}(2n-1)!c_0}{(\tau^2+16(t^2+u_1^2+u_2^2))^{n+1}}du_1du_2=\\&=\frac{16n}{\pi}\frac{4^{2n}(2n-1)!c_0}{16^{n+1}}\int\limits_{\mathbb{R}^2}\frac{1}{\left(\frac{\tau^2}{16}+t^2+(u_1^2+u_2^2)\right)^{n+1}}du_1du_2=\\&=\frac{16n}{\pi}\frac{4^{2n}(2n-1)!c_0}{16^{n+1}}\int\limits_{\frac{-\pi}{2}}^{\frac{3}{2}\pi}\int\limits_0^\infty\frac{\rho}{\left(\frac{\tau^2}{16}+t^2+\rho^2\right)^{n+1}}d\rho d\theta=\\&=\frac{4^{2n}(2n-1)!c_0}{(\tau^2+16|z|^2)^n},\end{aligned}$$

where $z=t\xi$. Let $\varphi(\tau,z)=\frac{4^{2n}(2n-1)!c_0}{(\tau^2+16|z|^2)^n}$. Now from the expression of the fundamental solution of L in the classical case (see for example 4.3 of [G-S(2)]) we know that

$$\varphi(\tau,t\xi)=\sum_{k\geq 0}\frac{(-1)}{2k+2n}\int\limits_{-\infty}^{\infty}e^{i\lambda t}L_k^{2n-1}\left(\frac{\lambda}{2}|\tau|\right)e^{-\frac{\lambda}{4}|\tau|}|\lambda|^{2n-1}d\lambda.$$

The first step to compute Φ is to change to polar coordinates in $\mathbb{R}^3$ the expression given in (1.4):

$$\begin{aligned}<\Phi,f>&=\sum_{k\in\mathbb{Z}}\int\limits_{\mathbb{R}^3}\frac{1}{-|\lambda|(2k+2(p-q))}<\varphi_{w,k},f>|w|^{2n}dw=\\&=\sum_{k\in\mathbb{Z}}\int\limits_{S^2}\int\limits_0^\infty\frac{1}{-|\lambda|(2k+2(p-q))}<\varphi_{\lambda\xi,k},f>|\lambda|^{2n+2}d\lambda d\xi.\end{aligned}$$

From the absolute convergence of (1.4) we can interchange the summation symbol with the integral

over S^2. Because of $\triangle e^{i\lambda<\xi,z>} = -|\lambda|^2 e^{i\lambda<\xi,z>}$, we have that

$$< \Phi, f >= \int_{S^2} \sum_{k\in\mathbb{Z}} \frac{(-1)}{(2k+2(p-q))} \int_0^\infty \int_{N(p,q,\mathbb{H})} e^{i\lambda<\xi,z>} \theta_{\lambda,k}(\alpha) f(\alpha,z) d\alpha dz |\lambda|^{2n+1} d\lambda d\xi =$$

$$= \int_{S^2} \sum_{k\in\mathbb{Z}} \frac{1}{(2k+2(p-q))} \int_0^\infty \int_{N(p,q,\mathbb{H})} \triangle e^{i\lambda<\xi,z>} \theta_{\lambda,k}(\alpha) f(\alpha,z) d\alpha dz |\lambda|^{2n-1} d\lambda d\xi =$$

$$= \int_{S^2} \sum_{k\in\mathbb{Z}} \frac{1}{(2k+2(p-q))} \int_0^\infty < \varphi_{\lambda\xi,k}, \triangle f > |\lambda|^{2n-1} d\lambda d\xi.$$

Next we break the summation indexes according to $k \geq 2q$, $k \leq -2p$ and $-2p < k < 2q$ to get the splitting $< \Phi, f >=< \Phi_1, f > + < \Phi_2, f >$, and as in section we change summation index to get the series starting from $k = 0$. From the explicit definition of $\varphi_{\lambda\xi,k}$ we can write

$$< \Phi_1, f >= \int_{S^2} \sum_{k\geq 0} \frac{1}{2k+2n} \int_0^\infty \int_{\mathbb{R}^3} e^{i\lambda<\xi,z>} \times$$
$$\times < T_{\lambda,k+2q} - T_{\lambda,-k-2p}, N\triangle f(.,z) > dz|\lambda|^{2n-1} d\lambda d\xi,$$

where $T_{\lambda,k}$ is defined by equations (1.6) and (1.7). By performing similar computations than in section 2 and introducing the function

$$G_f(\tau,z) = Nf(\tau,z) - \sum_{j=0}^{2n-2} \frac{\partial^j Nf}{\partial\tau^j}(0,z)\frac{\tau^j}{j!}$$

we get the splitting $< \Phi_1, f >=< \Phi_{11}, f > + < \Phi_{12}, f >$, where

$$< \Phi_{11}, f > = \int_{S^2} \sum_{k\geq 0} \frac{(-1)}{2k+2n} \int_0^\infty \int_{\mathbb{R}^3} \int_{-\infty}^\infty e^{i\lambda<\xi,z>} \times \tag{3.2}$$
$$\times \ \mathrm{sgn}(\tau) L_k^{2n-1}\left(\frac{2}{\lambda}|\tau|\right) e^{-\frac{\lambda}{4}|\tau|} \triangle G_f(\tau,z) d\tau dz |\lambda|^{2n-1} d\lambda d\xi,$$

$$< \Phi_{12}, f > = 2\int_{S^2} \sum_{k\geq 0} \frac{1}{2k+2n} \int_0^\infty \int_{\mathbb{R}^3} e^{i\lambda<\xi,z>} \times \tag{3.3}$$
$$\times \sum_{\substack{l=0 \\ l \text{ odd}}}^{2n-2} \left(\frac{2}{\lambda}\right)^{l+1} (a_{kl} + b_{kl}) < \delta^{(l)}, \triangle Nf(.,z) > dz|\lambda|^{2n-1} d\xi,$$

and a_{kl}, b_{kl} are the same constant defined in (2.8) and (2.3), respectively. Now let us recall the fact that

$$\int_{S^2} \int_0^\infty e^{i\lambda<\xi,z>} F(|\lambda|) d\lambda d\xi = \frac{1}{2} \int_{S^2} \int_{-\infty}^\infty e^{i\lambda<\xi,z>} F(|\lambda|) d\lambda d\xi,$$

and we apply the dual Radon transform to (3.2).

Observe now that

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{\operatorname{sgn}(\tau) G_f(\tau,z)}{(1+16|z|^2)^{n+1}} dz d\tau$$

converges, which we can see by changing to polar variables in $\mathbb{R}^3$ and arguing like in lemma 2.2 of [G-S(2)].

We finally get that

$$\begin{aligned} < \Phi_{11}, f >&= \frac{1}{2} < -2\pi \frac{16n}{\pi} \frac{4^{2n}(2n-1)! c_0}{(\tau^2+16|z|^2)^{n+1}},\ \operatorname{sgn}(\tau) G_f(\tau,z) >= \\ &= -4^{2n+2} n(2n-1)! c_0 < \frac{1}{(\tau^2+16|z|^2)^{n+1}},\ \operatorname{sgn}(\tau) G_f(\tau,z) > . \end{aligned}$$

We have thus proven that the expression defining Φ_{11} is finite. Then the expression defining Φ_{12} is also finite, and by Abel's lemma we can write

$$< \Phi_{12}, f > = 2 \lim_{r\to 1^-} \lim_{\epsilon \to 0^+} \sum_{k\geq 0} \sum_{\substack{l=0 \\ l \text{ odd}}}^{2n-2} 2^{l+1}(a_{kl}+b_{kl}) \frac{r^{2k+2n}}{2k+2n} d_{\epsilon,l,f},$$

where

$$d_{\epsilon,l,f} = \int_{S^2}\int_0^{\infty}\int_{\mathbb{R}^3} e^{-\epsilon\lambda} e^{i\lambda<\xi,z>} |\lambda|^{2n-l-2} < \delta^{(l)}, \triangle N f(.,z) > dz d\lambda d\xi. \tag{3.4}$$

We need to compute $\lim_{\epsilon\to 0^+} d_{\epsilon,l,f}$. Observing that $\triangle e^{i\lambda<\xi,z>} = -|\lambda|^2 e^{i\lambda<\xi,z>}$, we have that

$$\begin{aligned} d_{\epsilon,l,f} &= (-1)^{l+1} \int_{S^2}\int_0^{\infty}\int_{\mathbb{R}^3} e^{-\epsilon\lambda} e^{i\lambda<\xi,z>} |\lambda|^{2n-l} \frac{\partial^l}{\partial\tau^l} N f(0,z) dz d\lambda d\xi = \\ &= (-1)^{l+1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\epsilon|x|} |x|^{2n-l-2} e^{i<x,z>} \frac{\partial^l}{\partial\tau^l} N f(0,z) dz dx, \end{aligned}$$

where we have changed to cartesian coordinates in $\mathbb{R}^3$. To solve this integral let us observe that

$$(-1)^{2n-l-2} e^{-\epsilon|x|} |x|^{2n-l-2} = \frac{\widehat{\partial^{2n-l-2}}}{\partial\epsilon^{2n-l-2}} P_\epsilon(x),$$

where $P_\epsilon(x)$ is the Poisson kernel and $\hat{}$ denotes the Fourier transform. Let us write

$$d_{\epsilon,l,f} = (-1)^l \int_{\mathbb{R}^3} \frac{\widehat{\partial^{2n-l-2}}}{\partial\epsilon^{2n-l-2}} P_\epsilon(x) \left(\frac{\partial^l}{\partial\tau^l} N f(0,.)\right)^{\hat{}}(x) dx = (-1)^l \frac{\partial^{2n-l-2}}{\partial\epsilon^{2n-l-2}} (P_\epsilon * h)(0).$$

Taking limit as $\epsilon \to 0^+$ we obtain

$$\lim_{\epsilon\to 0^+} = (-1)(-\triangle)^{\frac{2n-l-2}{2}} \frac{\partial^l}{\partial\tau^l} N f(0,0),$$

where $(-\triangle)^{\frac{2n-l-2}{2}}$ is a fractionary exponential of the Laplacian (see for example [S-Sh]), which is the operator defined for $g \in \mathcal{S}(\mathbb{R}^3)$ by

$$(-\triangle)^{\frac{2n-l-2}{2}} g(x) = \int_{\mathbb{R}^3} |\omega|^{2n-l-2} \widehat{g}(\omega) e^{i<\omega,z>} d\omega.$$

This computation together with proposition 4.8 of [G-S(2)] lets us write

$$< \Phi_{12}, f >= \sum_{\substack{l=0 \\ l \text{ odd}}}^{2n-2} \sum_{j=1}^{l+1} \frac{1}{2^{2n-2l-j-3}} c_j \binom{2n-j-1}{l-j-1} (-1)(-\triangle)^{\frac{2n-l-2}{2}} \frac{\partial^l}{\partial \tau^l} Nf(0,0),$$

where each c_j is the constant defined in Remark 4.7 of [G-S(2)].

After performing the usual computations for Φ_2 we have proved the main theorem of this section:

Theorem 3.1. *Let c_0 and c_j be the constants defined above. Then*

$$\begin{aligned}
< \Phi, f >= &-4^{2n+2} n(2n-1)! c_0 < \frac{1}{(\tau^2+16|z|^2)^{n+1}},\ sgn(\tau) G_f(\tau, z) > + \\
&+ \sum_{\substack{l=0 \\ l odd}}^{2n-2} \sum_{j=1}^{l+1} \frac{1}{2^{2n-2l-j-3}} c_j \binom{2n-j-1}{l-j-1} (-1)(-\triangle)^{\frac{2n-l-2}{2}} \frac{\partial^l}{\partial \tau^l} Nf(0,0) + \\
&+ \sum_{-2q<k<0} \frac{1}{2k+2(p-q)} \Bigg[(-1)^{k+1} \sum_{r=0}^{k+2p-1} \binom{k+2p-1}{r} 2^{-k-2p+2r+2} \times \\
&\quad \times (-1)(-\triangle)^{\frac{2n-r-2}{2}} \frac{\partial^r}{\partial \tau^r} Nf(0,0) + \\
&+ \sum_{l=0}^{2n-2-k-2p} (-1)^{-l-k} \binom{-k-2p+2n-1}{2n-2-l-k-2p} \sum_{r=0}^{l+k-2p} \binom{l+k-2p}{r} \times \\
&\quad \times (-1) 2^{-l-k+2p+2r+1} (-\triangle)^{\frac{2n-r-2}{2}} \frac{\partial^r}{\partial \tau^r} Nf(0,0) \Bigg] + \\
&+ \sum_{0 \le k < 2q} \frac{1}{2k+2(p-q)} \Bigg[(-1)^{k+1} \sum_{r=0}^{-k+2q-1} \binom{-k+2q-1}{r} \times \\
&\quad \times 2^{k-2q+2r+1} (-1)^r (-\triangle)^{\frac{2n-r-2}{2}} \frac{\partial^r}{\partial \tau^r} Nf(0,0) + \\
&+ \sum_{l=0}^{2n-2+k-2q} (-1)^{-l+k-2q} \binom{k-2q+2n-1}{2n-2-l+k-2q} \sum_{r=0}^{l-k+2q} \binom{l-k+2q}{r} \times \\
&\quad \times 2^{-l+k-2q+2r+1} (-1)^r (-\triangle)^{\frac{2n-r-2}{2}} \frac{\partial^r}{\partial \tau^r} Nf(0,0) \Bigg].
\end{aligned}$$

Remark

Let N be a group of Heisenberg type and let η be its Lie algebra. So $\eta = V \oplus \mathfrak{z}$, with $\dim V = 2m$ and $\dim \mathfrak{z} = k$. Let $U(V)$ be the unitary group acting on V. Then it is known ([R]) that $(N \ltimes U(V), U(V))$

is a Gelfand pair, and also in [R] were computed the spherical functions. We fix an orthonormal basis of V, $\{X_1, \ldots, X_{2m}\}$, and consider the operator

$$L = \sum_{j=1}^{2m} X_j^2.$$

With the same arguments as above, using the Radon transform in $\mathbb{R}^k$ and the fundamental solution of L in the classical Heisenberg group $2m+1$ dimensional, we can recover the fundamental solution of L (see [K], [R]).

References

[]B-D-R C. Benson, A.H. Dooley and G. Ratcliff, *Fundamental solutions for powers of the Heisenberg sub-Laplacian*, Ill. J. Math. 37, No.3, 455-476 (1993).

[]D-M G. Van Dijk and K. Mokni, *Harmonic analysis on a class of generalized Gel'fand pairs associated with hyperbolic spaces*, Russ. J. Math. Phys. 5, No.2, 167-178 (1997).

[]F G.B. Folland *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. 79 (1973), 373-376.

[]F-S G.B. Folland and E.M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group.*, Commun. Pure Appl. Math. 27, 429-522 (1974).

[]G-S(1) T. Godoy, L. Saal, *L^2 spectral decomposition on the Heisenberg group associated to the action of $U(p,q)$*, Pac. J. Math. 193, No.2, 327-353 (2000).

[]G-S(2) T. Godoy, L. Saal, *On the relative fundamental solutions for a second order differential operator on the Heisenberg group*, Stud. Math. 145, No.2, 143-164 (2001).

[]K A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Am. Math. Soc. 258, 147-153 (1980).

[]K-R A. Kaplan and F. Ricci, *Harmonic analysis on groups of Heisenberg type*, Lecture notes on Math 992, 1983, 416-435.

[]M-R D. Müller and F. Ricci, *On the Laplace-Beltrami operator on the oscillator group*, J. Reine Angew. Math. 390, 193-207 (1988).

[]R F. Ricci, *Harmonic analysis on groups of type H*, preprint.

[]S-Sh E.M. Stein and R. Shakarchi, *Fourier Analysis:An Introduction*, Princeton University. Press, 2003.

[]T A. Tengstrand, *Distributions invariant under an orthogonal group of arbitrary signature*, Math. Scand. 8, 201-218 (1960).

[]V R. Vidal, *Anlisis armnico en el grupo de Heisenberg cuaterninico asociado al grupo* $Spin(3) \times U(p, q, \mathbb{H})$, Trabajos de Matemtica - FaMAF, 2009.